15. E. J. Kerschen, "Constraints on the invariant functions of axisymmetric turbulence," AIAA J., 21, No. 7 (1983).
16. J. Lamley, "Second-order methods for turbulent flows," in: Methods of Calculating Turbulent Flows [Russian translation], Mir, Moscow (1984).
17. H. K. Moffat, "The degree of knottedness of tangled vortex lines," J. Fluid Mech., 35, Part 1 (1969).
18. A. Levich and A. Tsinober, "On the role of helical structures in three-dimensional turbulent flow," Phys. Lett., 93A, No. 6 (1983).
19. H. K. Moffat, "Transport effects associated with turbulence, with particular attention to the influence of helicity," Rep. Prog. Phys., 46, No. 5 (1983).

PHENOMENOLOGICAL DESCRIPTION OF TWO-VELOCITY MEDIA WITH RELAXING TANGENTIAL STRESSES

V. N. Dorovskii and Yu. V. Perepechko

UDC 530.1

Tangential stresses are generated and subsequently relax during the filtration process of high-temperature solutions (or melts) through an enclosing island. The stresses generated as well as their relaxation dynamics start determining, in turn, the filtration mechanics of the fluid phase leading to a self-contained interaction process of the continua under consideration.

The concept of effective elastic deformation was suggested in [1] to describe the relaxation of tangential stresses is a viscoelastic medium. By introducing it the authors succeeded in generalizing the Maxwell relaxation model to the case of substantial medium deformation. This is one of the principal approaches in nonlinear filtration theory. The generalization of the Maxwell model to filtration media within the approximations of small deformations and low velocities of the filtering fluid was investigated numerous times in the literature (see, for example, [2]). To the best of the authors' knowledge, the extension of the Maxwell model to the case of nonlinear island deformation and high fluid filtration rates is not available in the literature.

Under conditions of filtration of a viscous fluid through viscoelastic medium the effective elastic deformation must be introduced somewhat differently than was done in [1]. A theory using the concept of effective elastic deformation must be compatible with general physical requirements: conservation laws and the Galileo relativity principle.

Below we obtain a system of differential equations, describing the relaxation of tangential stresses of a viscoelastic island during its self-consistent interaction with a filtering viscous fluid. The necessary requirement on the initial deformation of the state of the medium is established. The system of equations describes both compact and noncompact two-velocity continua.

For a basis of the general theory one must construct a formalism of elastic interaction of the island with the filtering fluid in the reversible hydrodynamic approximation. To describe the filtration process within the continuum approach we introduce two velocity fields: $\mathbf{u}$ – the velocity of motion of an elastic continuum with particle density ρ_1, and $\mathbf{v}$ – the velocity of motion of a fluid with partial density ρ_2, filtering through the elastic continuum. Two such mutually penetrable continua can interact through a friction force $\mathbf{f}$, which is not present in the reversible approximation, and a reaction force being in hydrodynamics proportional to gradients of thermodynamic quantities. Besides, the set of two continua is a hydrodynamic system for which conservation laws are valid, being in the case of reversible motion

Novosibirsk. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 3, pp. 97-104, May-June, 1992. Original article submitted July 12, 1990; revision submitted March 22, 1991.

$$\partial\rho/\partial t + \operatorname{div}\mathbf{j} = 0, \quad \partial j_i/\partial t + \nabla_j \Pi_{ij} = 0, \tag{1}$$
$$\partial S/\partial t + \operatorname{div}\mathbf{F} = 0, \quad \partial E/\partial t + \operatorname{div}\mathbf{Q} = 0.$$

Here the extensive quantities refer to a unit volume: ρ is the density of the continuum aggregate, $\mathbf{j}$ is the momentum, Π_{ij} is the tensor of momentum flux, $\mathbf{F}$ is the generating entropy flux, $\mathbf{Q}$ is the reversible energy flux, E is the energy, S is the entropy, and T is the temperature.

The equations of motion of a filtering fluid are taken in a form generalizing the Euler equations and compatible with the conditions of thermodynamic equilibrium μ = const, T = const, $\mathbf{u} = \mathbf{v} = 0$ for the composite hydrodynamic system

$$\partial\mathbf{v}/\partial t + (\mathbf{v}, \nabla)\mathbf{v} = -\nabla\mu - \alpha\nabla\Omega, \tag{2}$$

where $\alpha\nabla\Omega$ is the density of bulk forces, and μ is the chemical potential. In the following Eqs. (1), (2) are supplemented to form a complete system of equations.

The system (1), (2) is redetermined since the energy conservation law is not independent in hydrodynamics. The result of compatibility of the system of equations is to obtain $\mathbf{j}$, Π_{ij}, $\mathbf{F}$, $\mathbf{Q}$, Ω, as well as the relations between them. A constructive mechanism of this approach is the Galileo relativity principle [3]. Indeed, we transform to a coordinate system in which the fluid phase is at rest. The physical quantities referring to this system are denoted by the subscript zero. In this case we have [3]

$$E = \rho v^2/2 + (\mathbf{v}, \mathbf{j}_0) + E_0, \quad \mathbf{j} = \rho\mathbf{v} + \mathbf{j}_0. \tag{3}$$

In the selected reference system $\mathbf{j}_0 = \rho_1(\mathbf{u} - \mathbf{v})$, while the first law of thermodynamics is

$$dE_0 = TdS + \mu d\rho + (\mathbf{u} - \mathbf{v}, d\mathbf{j}_0) + (1/2)h_{\alpha\beta}dg^{\alpha\beta} \tag{4}$$

($g^{\alpha\beta}$ is the metric deformation tensor, and $h_{\alpha\beta}$ is the "stress" tensor). The third term in (4) reflects the presence of motion in the continuum element and is impossible to remove by the selected system of reference. The last term takes into account the energy of elastic deformation. We note that the true stress tensor is found from the expression

$$-\sigma_{ik} = p\delta_{ik} + h_{\alpha\beta}e_i^\alpha e_k^\beta.$$

Describing the thermodynamic state of the continuum element by the variables S, ρ, $g^{\alpha\beta}$ in a reference system attached to the resting fluid particles, we allow the presence of relative motion in the selected continuum element, which at the same time is described by an additional thermodynamic variable – the relative momentum $\mathbf{j}_0$. Thus, we have a locally nonequilibrium thermodynamic system with relaxing degrees of freedom, for which one can select the components of $\mathbf{j}_0$. According to Leontovich Eq. (4) determines the locally nonequilibrium entropy as a function of system energy, external parameters, and relaxing degrees of freedom ($\xi_i \to j_{0,i}$) [4]:

$$dS = \frac{dE_0}{T} - \frac{\mu}{T}d\rho - \frac{(\mathbf{u} - \mathbf{v}, d\mathbf{j}_0)}{T} - \frac{1}{2}\frac{h_{\alpha\beta}}{T}dg^{\alpha\beta}.$$

It is precisely in this sense that (4) must be understood.

Differentiating the first of relations (3) with respect to time, with account of (4) we obtain the rate of energy change as a function of spatial derivatives of thermodynamic quantities:

$$\frac{\partial E}{\partial t} = \left(\mu + \frac{v^2}{2} - (\mathbf{u}, \mathbf{v})\right)\frac{\partial\rho}{\partial t} + T\frac{\partial S}{\partial t} + \left(\mathbf{j} - \rho\mathbf{u}, \frac{\partial\mathbf{v}}{\partial t}\right) + \left(\mathbf{u}, \frac{\partial\mathbf{j}}{\partial t}\right) + \tag{5}$$
$$+ \frac{1}{2}h_{\alpha\beta}\frac{\partial g^{\alpha\beta}}{\partial t}.$$

The last derivative can be expressed in terms of the local n-hedral $\{\mathbf{e}^\alpha\}$, characterizing the deformation state: $g^{\alpha\beta} = (\mathbf{e}^\alpha, \mathbf{e}^\beta)$, whose vectors satisfy the system of dynamic equations [5]

$$\partial\mathbf{e}^\alpha/\partial t + \nabla(\mathbf{u}, \mathbf{e}^\alpha) = 0. \tag{6}$$

Replacing in (5) the time derivatives, and collecting then the spatial derivatives under the divergence sign, we reach the energy conservation law

$$\partial E/\partial t + \operatorname{div}\left([\mu + v^2/2 + ST/\rho]\mathbf{j} + \mathbf{u}(\mathbf{u}, \mathbf{j}_0) + h_{\alpha\beta}\mathbf{e}^{\alpha}(\mathbf{e}^{\beta}, \mathbf{u})\right) = 0,$$

thus determining $\mathbf{Q}$. At the same time we find $\mathbf{F} = (S/\rho)\mathbf{j}$, $\rho = \rho_1 + \rho_2$, $\alpha = -S/\rho$, $\Omega = T$, $\Pi_{ik} = \rho v_i v_k + j_{0,k}v_i + u_k j_{0,i} + p\delta_{ik} + h_{\alpha\beta}e_i^{\alpha}e_k^{\beta}$ as conditions of representing the energy conservation law in divergence form. The procedure of transforming the spatial derivatives under the divergence sign is discussed in [5] quite in detail.

The pressure is given in the standard way: $p = -E_0 + TS + \mu\rho + (\mathbf{u} - \mathbf{v}, \mathbf{j}_0)$, which together with (4) makes it possible to write down the Gibbs–Duhem identity

$$dp = \rho d\mu + SdT + \mathbf{j}_0 d(\mathbf{u} - \mathbf{v}) - (1/2)h_{\alpha\beta}dg^{\alpha\beta}. \tag{7}$$

To sum up, with account of (6) the equation of motion (2) acquires the form [6]

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\rho_1}{2\rho}\nabla(\mathbf{u} - \mathbf{v})^2 - \frac{1}{2\rho}h_{\alpha\beta}\nabla g^{\alpha\beta}. \tag{8}$$

Equations (1), (6), (8) with the inverse fluxes found $\mathbf{j}$, Π_{ij}, $\mathbf{F}$ and the given equation of state $E_0 = E_0(\rho, S, \mathbf{j}_0, g^{\alpha\beta})$ describe completely reversible filtration in the isotropic system under consideration. We note that it is not necessary to supplement the system of equations by the conservation law $\partial\rho_1/\partial t + \operatorname{div}(\rho_1 \mathbf{u}) = 0$, since, if $\rho_1 \sim 1/\sqrt{g}$ ($g = \det(g_{\alpha\beta})$ according to nonlinear elasticity theory [1] the "conservation" of the solid component is a consequence of system (6). For $\mathbf{j}_0 = 0$ the partial densities ρ_1 and ρ_2 are easily related to the volume fraction x of the fluid component (with $k_{\alpha\beta} = 0$):

$$\rho = \rho_1 + \rho_2 = \rho_1^f(1 - x) + \rho_2^f x,$$

where ρ_1^f, ρ_2^f are the physical densities of the components. The kinetic corrections to ρ_1 and ρ_2, related to $\mathbf{j}_0 \neq 0$ within the quadratic approximation, are easily found by using identity (7) and a relation introducing the bulk fraction of the fluid component, similarly to (3).

The system of nonlinear hydrodynamic equations (1), (6), (8) describes, within the reversible approximation, filtration for arbitrary velocity values in the elastically deformed island. Omitting in (4) the energy of elastic deformations, i.e., making the components hydrodynamically uniform in the system (1), (6), (8), then the pair of equations and the momentum conservation law of system (1) is replaced by the equivalent system

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\rho_1}{2\rho}\nabla(\mathbf{u} - \mathbf{v})^2,$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}, \nabla)\,\mathbf{u} = -\frac{1}{\rho}\nabla p - \frac{\rho_2}{2\rho}\nabla(\mathbf{u} - \mathbf{v})^2.$$

One easily notes the limiting transition to the single-velocity continuum for $\mathbf{u} \to \mathbf{v}$:

$$\partial \mathbf{v}/\partial t + (\mathbf{v}, \nabla)\mathbf{v} = -(1/\rho)\nabla p.$$

As could be expected, when the component motion is controlled by hydrodynamically uniform equations, the limiting transition $\mathbf{u} \to \mathbf{v}$ to the Euler equation is implemented. We note the problematic feature of this limiting transition even in the equations of linear filtration theory (see, for example, [7, p. 240]).

The equations provided of nonlinear filtration theory, obtained by well-known continuum methods, are widely used in two-velocity hydrodynamics of superfluid helium [3]. The methodology of deriving the equations of motion differs from that of averaging methods (see, for example, [8]) and has, it seems to us, a more general character. In studies related to averaging methods one puts

$$E = E_{0,1} + \rho_1 u^2/2 + E_{0,2} + \rho_2 v^2/2,$$

where $E_{0,1}$ and $E_{0,2}$ is the internal energy of components, being independent of u, v. In other words, interaction forces are not being considered. In the general case one cannot select a reference system in which one could eliminate the relative motion of components, implying that the energy of the whole system cannot be divided into internal and kinetic. Separating the same energy into two parts, interaction forces, which in the reversible approximation are reduced to d'Alembert reaction forces [9], are omitted from the treatment, which distinguishes our theory substantially from the corresponding constructions of other authors. As a consequence of this simplification in the equations of motion the terms $(\rho_1/2\rho)\,\Delta(\mathbf{u}-\mathbf{v})^2$, $(\rho_2/2\rho)\,\Delta(\mathbf{u}-\mathbf{v})^2$ vanish, and problems arise of the limiting transition to a single-velocity continuum (even within the linear approximation). At the same time we note that the supplementary terms in the equations of motion are quadratic in velocity, along with the terms $(\mathbf{v},\nabla)\mathbf{v}$ and $(\mathbf{u},\nabla)\mathbf{u}$. Therefore, their simultaneous presence in the equations of motion is of principal importance. This is a substantial difference between the system obtained and the system of linear equations derived, for example, in [10]. As will be shown below, dissipative forces are obtained within the irreversible approximation according to the general principles of thermodynamics of irreversible processes. Their shape does not require any additional assumptions, unlike studies in filtration theory, based on the averaging of "microscopic" equations of motion.

Before turning to describe hydrodynamic relaxation with entropy production, Eq. (6) is given in the reversible case in equivalent form, making it possible to generalize to the case of irreversible motion. According to (6), curl $\mathbf{e}^\alpha = 0$ or $\partial_i e^\alpha_k - \partial_k e^\alpha_i = 0$. The last relation is conveniently "convoluted" with velocity $\mathbf{u}$ and combined with (6). As a result we reach an equation equivalent to the reversible approximation (6):

$$\partial e^\alpha_i/\partial t + \partial_i\left(u_k e^\alpha_k\right) = K^\alpha_{ik} u_k$$

($K^\alpha_{ik} = \partial_i e^\alpha_k - \partial_k e^\alpha_i$). In the general case the irreversible approximation is not compatible with the requirement curl $\mathbf{e}^\alpha = 0$.

Irreversible thermodynamic relaxation can be described by supplementing the reversible fluxes with the corresponding irreversible fluxes and introducing the required sources, whose shape is determined later:

$$\partial e^\alpha_i/\partial t + \partial_i\left(u_k e^\alpha_k + \varphi_k e^\alpha_k\right) = \psi_{ik} e^\alpha_k + K^\alpha_{ik}(u_k + \chi_k); \tag{9}$$

$$\partial j_i/\partial t + \partial_k\left(\Pi_{ik} + \pi^{(u)}_{ik} + \pi^{(v)}_{ik}\right) = 0; \tag{10}$$

$$\frac{\partial S}{\partial t} + \operatorname{div}\left(S\frac{\mathbf{j}}{\rho} + \frac{\mathbf{q}}{T}\right) = \frac{R}{T}; \tag{11}$$

$$\partial\rho/\partial t + \operatorname{div}\mathbf{j} = 0; \tag{12}$$

$$\partial\mathbf{v}/\partial t + (\mathbf{v},\nabla)\mathbf{v} = -\nabla(\mu + h) - (S/\rho)\nabla T + \mathbf{f}; \tag{13}$$

$$\partial E/\partial t + \operatorname{div}(\mathbf{Q} + \mathbf{W}) = 0. \tag{14}$$

Here $\boldsymbol{\varphi}$, $\boldsymbol{\chi}$, $\mathbf{q}/T$, $\mathbf{W}$ are irreversible vector fluxes ψ_{ik}, $\pi^{(u,v)}_{ik}$ are irreversible tensor fluxes, h is a scalar flux, $\mathbf{f}$ is the force of intercomponent friction, and R is the dissipative function.

We note that Eq. (9) has no solution of the form $e^\alpha_i = \partial_i F^\alpha$. Solutions of Eq. (9) by means of the relations $g^{\alpha\beta} = (\mathbf{e}^\alpha, \mathbf{e}^\beta)$ form the tensor of effective elastic deformations [1]. The energy conservation law (14) must follow identically from the system (9)-(13). As a result, all irreversible fluxes must be found uniquely. For this it is necessary to repeat the consistency algorithm of the system of equations discussed above. At the phase of separating reversible fluxes we reach

$$\frac{\partial E}{\partial t} + \operatorname{div}\left(\left[\mu + \frac{v^2}{2} + \frac{ST}{\rho}\right]\mathbf{j} + \mathbf{u}(\mathbf{u},\mathbf{j}_0) + h_{\alpha\beta}\mathbf{e}^\alpha(\mathbf{e}^\beta,\mathbf{u})\right) = R - \tag{15}$$

$$- T\operatorname{div}\left(\frac{\mathbf{q}}{T}\right) - u_i\partial_k\left(\pi^{(u)}_{ik} + \pi^{(v)}_{ik}\right) + (\mathbf{j} - \rho\mathbf{u})(\mathbf{f} - \nabla h) +$$

$$+ \frac{1}{2}h_{\alpha\beta}\left(\frac{\partial g^{\alpha\beta}}{\partial t} + (\mathbf{u},\nabla)g^{\alpha\beta}\right) + h_{\alpha\beta}e^\alpha_i e^\beta_k \partial_k u_i.$$

The last structure consisting of the two terms in (15) can be calculated by using the definition of the metric tensor and (9):

$$\frac{1}{2} h_{\alpha\beta} \frac{\partial g^{\alpha\beta}}{\partial t} + \frac{1}{2} h_{\alpha\beta} (\mathbf{u}, \nabla) g^{\alpha\beta} = h_{\alpha\beta} e_i^\alpha e_k^\beta (\psi_{ik} - \partial_i u_k - \partial_i \varphi_k) + \chi_k h_{\alpha\beta} e_i^\alpha K_{ik}^\beta - \\ - \varphi_k h_{\alpha\beta} e_i^\alpha \partial_i e_k^\beta.$$

Taking $g = \det(g_{\alpha\beta})$, then $1/\sqrt{g}$ is conveniently related to the partial density $\rho_1 \sim 1/\sqrt{g}$ of the viscoelastic continuum [1]. Since $dg = -g\, g_{\alpha\beta} dg^{\alpha\beta}$, then

$$\partial\rho_1/\partial t + \operatorname{div}(\rho_1(\mathbf{u} + \boldsymbol{\varphi})) = \rho_1(\delta_{ik} - \\ - (e_\alpha)_i (e^\alpha)_k)(\partial_i \varphi_k + \partial_i u_k) + L_{ik} (e_\alpha)_i (e^\alpha)_k.$$

Here the tensor L_{ik} is defined by the expression

$$L_{ik} e_k^\alpha = \psi_{ik} e_k^\alpha + K_{ik}^\alpha (\chi_k - \varphi_k).$$

The relations derived transform the right hand side of Eq. (15):

$$\frac{\partial E}{\partial t} + \operatorname{div} \mathbf{Q} = R - T \operatorname{div}\left(\frac{\mathbf{q}}{T}\right) - u_i \partial_k \pi_{ik}^{(u)} - v_i \partial_k \pi_{ik}^{(v)} + \\ + \left(f_i + \frac{1}{\rho_2} \partial_k \pi_{ik}^{(v)}\right)(j_i - \rho u_i) - (j_i - \rho u_i)\, \partial_i h - \partial_i \left(h_{\alpha\beta} e_i^\alpha e_k^\beta \varphi_k\right) + h_{\alpha\beta} e_i^\alpha e_k^\beta L_{ik} + \\ + \varphi_i \left(e_i^\beta \partial_k \left(h_{\alpha\beta} e_k^\alpha\right) + h_{\alpha\beta} e_k^\beta K_{ik}^\alpha\right).$$

Keeping in mind the two conditions (in the limit $\mathbf{u} \to \mathbf{v}$ and for $g^{\alpha\beta} = 0$ the theory must lead to the Navier–Stokes equations; phenomenological account of relaxation of tangential stresses leads to the nonlinear generalization of the Maxwell relaxation model), we obtain an expression for the irreversible part of the energy flux

$$\partial E/\partial t + \operatorname{div}\left(\mathbf{Q} + \mathbf{q} + h(\mathbf{j} - \rho\mathbf{u}) + \pi_{ik}^{(u)} u_k + \pi_{ik}^{(v)} v_k + h_{\alpha\beta} e_i^\alpha e_k^\beta \varphi_k\right) = 0.$$

The dissipation function acquires the following form under these conditions

$$-R = \frac{q_k}{T} \partial_k T + \left(f_i + \frac{1}{\rho_2} \partial_k \pi_{ik}^{(v)}\right)(j_i - \rho u_i) + \varphi_i \left(e_i^\beta \partial_k \left(h_{\alpha\beta} e_k^\alpha\right) + h_{\alpha\beta} e_k^\beta K_{ik}^\alpha\right) + \\ + h_{\alpha\beta} e_i^\alpha e_k^\beta L_{ik} + \pi_{ik}^{(u)} \partial_k u_i + \pi_{ik}^{(v)} \partial_k v_i + h \operatorname{div}(\mathbf{j} - \rho\mathbf{u}).$$

In the fluxes L_{ik}, $\pi_{ik}^{(u)}$, $\pi_{ik}^{(v)}$ it is convenient to separate the diagonal parts

$$\pi_{ik}^{(u)} = A_{ik} + a\delta_{ik}, \quad \pi_{ik}^{(v)} = B_{ik} + b\delta_{ik}, \quad L_{ik} = L_{ik}^* + L\delta_{ik}.$$

The notations introduced lead to the expression

$$-R = \frac{q_k}{T} \partial_k T + \left(f_i + \frac{1}{\rho_2} \partial_k \pi_{ik}^{(v)}\right)(j_i - \rho u_i) + \varphi_i \left(e_i^\beta \partial_k \left(h_{\alpha\beta} e_k^\alpha\right) + h_{\alpha\beta} e_k^\beta K_{ik}^\alpha\right) + \\ + a \operatorname{div}\mathbf{u} + b \operatorname{div}\mathbf{v} + h \operatorname{div}(\mathbf{j} - \rho\mathbf{u}) + 3L h_\nu^\nu + \frac{1}{2} A_{ik}\left(\partial_k u_i + \partial_i u_k - \frac{2}{3}\delta_{ik} \operatorname{div}\mathbf{u}\right) + \\ + \frac{1}{2} B_{ik}\left(\partial_k v_i + \partial_i v_k - \frac{2}{3}\delta_{ik} \operatorname{div}\mathbf{v}\right) + L_{ik}^* \left(e_i^\alpha e_k^\beta H_{\alpha\beta}\right).$$

Here $h_{\alpha\beta} = H_{\alpha\beta} + g_{\alpha\beta} h_\nu^\nu$. For scalar fluxes and forces we have the linear thermodynamic relations

$$-a = \zeta_{11} \operatorname{div}\mathbf{u} + \zeta_{12} \operatorname{div}\mathbf{v} + \zeta_{13} \operatorname{div}(\mathbf{j} - \rho\mathbf{u}) + \zeta_{14} h_\nu^\nu,$$
$$-b = \zeta_{21} \operatorname{div}\mathbf{u} + \zeta_{22} \operatorname{div}\mathbf{v} + \zeta_{23} \operatorname{div}(\mathbf{j} - \rho\mathbf{u}) + \zeta_{24} h_\nu^\nu,$$
$$-h = \zeta_{31} \operatorname{div}\mathbf{u} + \zeta_{32} \operatorname{div}\mathbf{v} + \zeta_{33} \operatorname{div}(\mathbf{j} - \rho\mathbf{u}) + \zeta_{34} h_\nu^\nu,$$
$$-L = \zeta_{41} \operatorname{div}\mathbf{u} + \zeta_{42} \operatorname{div}\mathbf{v} + \zeta_{43} \operatorname{div}(\mathbf{j} - \rho\mathbf{u}) + \zeta_{44} h_\nu^\nu.$$

To reveal the physical meaning of the flux L we calculate $(e_\alpha)_i e_k^\alpha$. We first show that the object $e_i{}^\alpha$ can be selected so as to satisfy $(e_\alpha)_i e_k^\alpha = \delta_{ik}$. Let $R_{ik} = g_{\alpha\beta} e_i^\alpha e_k^\beta$. Differentiating R_{ik} with respect to time, we obtain

$$\frac{\partial R_{ik}}{\partial t} = (e_\alpha)_i (e_\beta)_k \frac{\partial g^{\alpha\beta}}{\partial t} + (e_\alpha)_k \frac{\partial e_i^\alpha}{\partial t} + (e_\alpha)_i \frac{\partial e_k^\alpha}{\partial t}.$$

Using the equations for the corresponding derivatives gives

$$\partial R_{ik}/\partial t = (R_{k\nu} L_{i\nu} + R_{i\nu} L_{k\nu} - 2R_{i\nu} R_{k\mu} L_{\nu\mu}) + (R_{i\nu} R_{k\mu} + R_{i\mu} R_{k\nu} - R_{k\mu}\delta_{i\nu} - R_{i\mu}\delta_{k\nu})(\partial_\nu u_\mu + \partial_\nu \varphi_\mu) + (u_\nu + \varphi_\nu)\big((e_\alpha)_k \partial_\nu e_\mu^\alpha (R_{i\mu} - \delta_{i\mu}) + (e_\alpha)_i \partial_\nu e_\mu^\alpha (R_{k\mu} - \delta_{k\mu})\big).$$

The equation obtained for an arbitrary moment of time has the solution

$$R_{ik} = (e_\alpha)_i e_k^\alpha = \delta_{ik}. \tag{16}$$

It is necessary to require that condition (16) be satisfied at the initial moment of time. This is not difficult to do by assuming that the initial deformation state of the medium by continuous deformation of the metric in Euclidean space. It must be noted that in the presence of fluxes φ_i, ψ_{ik} the metric $g^{\alpha\beta}$ is generally speaking, non-Euclidean. The choice $R_{ik} = \delta_{ik}$ leads to the continuity equation $\partial\rho_1/\partial t + \mathrm{div}(\rho_1(\mathbf{u} + \boldsymbol{\varphi})) = 3\rho_1 L$. Thus, the scalar flux L describes the behavior of so-called noncompact bodies [1], whose irreversible deformation leads to the disappearance of cavities contained in them initially. For compact bodies it is necessary to put $\zeta_{14} = \zeta_{24} = \zeta_{34} = \zeta_{44} = 0$. For vector fluxes and forces we have the relations of linear thermodynamics of irreversible processes

$$-q_i = \alpha_{11}(1/T)\partial_i T + \alpha_{12}(j_i - \rho u_i) + \alpha_{13}\big(e_i^\beta \partial_k (h_{\alpha\beta} e_k^\alpha) + h_{\alpha\beta} e_k^\beta K_{ik}^\alpha\big),$$
$$-f_i = \frac{1}{\rho_2}\partial_k \pi_{ik}^{(v)} + \alpha_{21}\frac{1}{T}\partial_i T + \alpha_{22}(j_i - \rho u_i) + \alpha_{23}\big(e_i^\beta \partial_k (h_{\alpha\beta} e_k^\alpha) + h_{\alpha\beta} e_k^\beta K_{ik}^\alpha\big),$$
$$-\varphi_i = \alpha_{31}(1/T)\partial_i T + \alpha_{32}(j_i - \rho u_i) + \alpha_{33}\big(e_i^\beta \partial_k (h_{\alpha\beta} e_k^\alpha) + h_{\alpha\beta} e_k^\beta K_{ik}^\alpha\big).$$

In the absence of an irreversible flux φ_i it is necessary to put $\alpha_{3i} = 0$. Finally, the tensor fluxes A_{ik}, B_{ik}, L_{ik}, characterizing irreversible deformation, are expressed in terms of their corresponding thermodynamic forces:

$$-A_{ik} = \eta_{11}(\partial_k u_i + \partial_i u_k - (2/3)\delta_{ik}\,\mathrm{div}\,\mathbf{u}) + \eta_{12}(\partial_k v_i + \partial_i v_k - (2/3)\delta_{ik}\,\mathrm{div}\,\mathbf{v}) + \eta_{13} e_i^\alpha e_k^\beta H_{\alpha\beta},$$
$$-B_{ik} = \eta_{21}(\partial_k u_i + \partial_i u_k - (2/3)\delta_{ik}\,\mathrm{div}\,\mathbf{u}) + \eta_{22}(\partial_k v_i + \partial_i v_k - (2/3)\delta_{ik}\,\mathrm{div}\,\mathbf{v}) + \eta_{23} e_i^\alpha e_k^\beta H_{\alpha\beta},$$
$$-L_{ik}^* = \eta_{31}(\partial_k u_i + \partial_i u_k - (2/3)\delta_{ik}\,\mathrm{div}\,\mathbf{u}) + \eta_{32}(\partial_k v_i + \partial_i v_k - (2/3)\delta_{ik}\,\mathrm{div}\,\mathbf{v}) + \eta_{33} e_i^\alpha e_k^\beta H_{\alpha\beta}.$$

Thus, we reach the equations of motion

$$\partial j_i/\partial t + \partial_k(\rho_1 u_i u_k + \rho_2 v_i v_k + p\delta_{ik} + h_{\alpha\beta} e_i^\alpha e_k^\beta + A_{ik} + B_{ik} + a\delta_{ik} + b\delta_{ik}) = 0, \tag{17}$$
$$\partial S/\partial t + \mathrm{div}\,(S\mathbf{j}/\rho + \mathbf{q}/T) = R/T,\ \partial\rho/\partial t + \mathrm{div}\,\mathbf{j} = 0,$$

$$\frac{\partial v_i}{\partial t} + (\mathbf{v}, \nabla) v_i = -\frac{1}{\rho}\partial_i p + \frac{\rho_1}{2\rho}\partial_i(\mathbf{u} - \mathbf{v})^2 - \frac{1}{2\rho} h_{\alpha\beta}\partial_i g^{\alpha\beta} - \partial_i h - \frac{1}{\rho_2}\partial_k(B_{ik} + b\delta_{ik}) - \alpha_{21}\frac{1}{T}\partial_i T - \alpha_{22}(j_i - \rho u_i) - \alpha_{23}\big(e_i^\beta \partial_k (h_{\alpha\beta} e_k^\alpha) + h_{\alpha\beta} e_k^\beta K_{ik}^\alpha\big),$$
$$\partial e_i^\alpha/\partial t + \partial_i(u_k e_k^\alpha + \varphi_k e_k^\alpha) = L_{ik}^* e_k^\alpha + K_{ik}^\alpha (u_k + \varphi_k) + L e_i^\alpha,$$

which, including the equation of state $E_0 = E_0(\rho, S, j_0, g^{\alpha\beta})$, form a closed system, describing the filtration process of a viscous fluid through a viscoelastic medium. Following the method of [11], it is not difficult to obtain the linear version of the relaxation model. The phenomenologically constructed model describes the relaxation of tangential stresses in the island, through which filtration of the Newtonian fluid occurs. The theory is based on general physical principles in which case the Darcy relation is not used, unlike the widely used approach of [2]. This makes it possible to subsequently include nonlinear effects of filtration theory for arbitrary values of hydrodynamic velocities.

The approach used has made it possible to reveal the phenomenological nature of Darcy's relation, which, in turn, follows from system (17) by omitting inertial nonlinear terms in the velocity, as well as all kinetic coefficients besides α_{22}. We reach the system of equations

$$\partial_k\left(p\delta_{ik} + h_{\alpha\beta}e_i^\alpha e_k^\beta\right) = 0,$$

$$\frac{1}{\rho}\partial_i p + \frac{1}{2\rho}h_{\alpha\beta}\partial_i g^{\alpha\beta} + \alpha v_i = 0,\ \operatorname{div}(\rho_2 \mathrm{v}) = 0,$$

describing the interaction of the field v with the stress field $h_{\alpha\beta}$ in the absence of motion of the solid component. We note that the approach developed can be generalized to the case in which the filtering fluid manifests properties of a Maxwellian body. To describe then the fluid phase it is necessary to follow the approach adopted in this study for describing a viscoelastic island. Obtaining the original equations is not difficult when adopting our approach. The study of filtration systems, in which the very filtering fluid manifests properties different than a Newtonian fluid, is of decisive value in petroleum technology at the present time [12]. The approach developed becomes interesting for this class of problems, since Darcy's relation is not used in it.

LITERATURE CITED

1. S. K. Godunov, Elements of Mechanics of Continuous Media [in Russian], Nauka, Moscow (1978).
2. G. I. Barenblatt, V. M. Entov, and V. M. Ryshik, Theory of Nonstationary Filtration of Fluids and Gases [in Russian], Nedra, Moscow (1972).
3. L. D. Landau and E. M. Lifshitz, Fluid Mechanics, Pergamon Press (1989).
4. M. A. Leontovich, Introduction to Thermodynamics – Statistical Physics [in Russian], Nauka, Moscow (1986).
5. A. F. Andreev and Yu. M. Kagan, "Hydrodynamics of a rotating superfluid liquid," Zh. Eksp. Teor. Fiz., 86, No. 2 (1984).
6. V. N. Dorovskii, "Continual filtration theory," Geol. Geofiz., No. 7 (1989).
7. R.I. Nigmatulin, Foundations of Mechanics of Heterogeneous Media [in Russian], Nauka, Moscow (1978).
8. R. I. Nigmatulin, Dynamics of Multiphase Media [in Russian], Nauka, Moscow (1987), Parts 1, 2.
9. L. D. Landau and E. M. Lifshitz, Mechanics, Pergamon Press (1960).
10. P. H. Roberts and D. E. Loper, "Dynamical processes in slurries," in: Structure and Dynamics of Partially Solidified Systems, Boston (1987).
11. V. N Dorovskii and Yu. V. Perepechko, "Rheology of filtration systems," in: Embedding Theorems and Their Applications to Problems of Mathematical Physics [in Russian], Inst. Mat. Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1989).
12. V. G. Levashkevich, Nonlinear Effects in Fluid Filtration in a Porous Medium [in Russian], Nauk. Tekh., Minsk (1987).